

ZERO POINTS OF KILLING VECTOR FIELDS, GEODESIC ORBITS, CURVATURE, AND CUT LOCUS

BY

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ABSTRACT. Let (M, g) be a compact, connected, Riemannian manifold. Let X be a Killing vector field on M . $f = g(X, X)$ is called the length function of X . Let D denote the minimum of the distances from points to their cut loci on M . We derive an inequality involving f which enables us to prove facts relating D , the zero points of X , orbits of X which are closed geodesics, and, applying theorems of Klingenberg, the curvature of M . Then we use these results together with a further analysis of f to describe the nature of a Killing vector field in a neighborhood of an isolated zero point.

1. An inequality.

Theorem 1. Let X be a Killing vector field on M . Let q be a critical point of the length function $f = g(X, X)$ of X such that $f(q) \neq 0$. Assume the orbit γ of X through q is closed. Let a be another point of M and suppose the distance from q to a is ρ . Then we have

$$(1) \quad (\sqrt{f(q)} - \sqrt{f(a)})D/\sqrt{f(q)} \leq 2\rho.$$

Proof. Denote by β the period of the orbit γ . We note that γ is a geodesic, since q is a critical point of f [2, p. 356]. Let τ be the orbit of X through a . Now assume $(\sqrt{f(q)} - \sqrt{f(a)})D/\sqrt{f(q)} > 2\rho$. Pick an integer m and a real number r so that

$$(2) \quad m\beta - r = D/\sqrt{f(q)} - \delta$$

where $\delta > 0$ is chosen sufficiently small so that $(\sqrt{f(q)} - \sqrt{f(a)})(D/\sqrt{f(q)} - \delta) > 2\rho$. Then we have

$$(3) \quad \sqrt{f(q)}(m\beta - r) - \sqrt{f(a)}(m\beta - r) > 2\rho.$$

Let $d(v, w)$ denote the distance between two points v and w in M . Let ϕ_t be the flow of X . We have $\phi_{m\beta-r}(q) = \phi_{m\beta}(\phi_{-r}(q)) = \phi_{-r}(q)$ since m is an integer and β is the period of γ . The length of the shortest segment of γ between q and $\phi_{-r}(q)$ is $\int_0^{m\beta-r} \sqrt{f(\phi_t(q))} dt$, which is equal to $\sqrt{f(q)}(m\beta - r)$ since f is constant along γ . This is the length of the shortest segment because

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by (2), $\sqrt{f(q)}(m\beta - r) = D - \delta\sqrt{f(q)} < D$ and clearly $D \leq \frac{1}{2}$ (length of γ). This is also the length of the shortest segment of γ between q and $\phi_r(q)$. Since γ is a geodesic and since the length of the shortest segment of γ between q and $\phi_r(q)$ is less than D , this segment is therefore length minimizing. Hence we have

$$(4) \quad d(q, \phi_r(q)) = \sqrt{f(q)}(m\beta - r).$$

Since f is constant along the orbit τ of X through a , we have that the length of a segment of τ between $\phi_{m\beta}(a)$ and $\phi_r(a)$ is $\sqrt{f(a)}(m\beta - r)$. Thus we have

$$(5) \quad d(\phi_{m\beta}(a), \phi_r(a)) \leq \sqrt{f(a)}(m\beta - r).$$

From (3), (4), and (5) we obtain $d(q, \phi_r(q)) > 2\rho + \sqrt{f(a)}(m\beta - r) \geq 2\rho + d(\phi_{m\beta}(a), \phi_r(a))$. Hence

$$(6) \quad d(q, \phi_r(q)) > 2\rho + d(\phi_{m\beta}(a), \phi_r(a)).$$

By the triangle inequality we obtain

$$d(q, \phi_r(a)) \leq d(q, \phi_{m\beta}(a)) + d(\phi_{m\beta}(a), \phi_r(a)).$$

But $d(q, \phi_{m\beta}(a)) = d(\phi_{m\beta}(q), \phi_{m\beta}(a)) = d(q, a) = \rho$ since $\phi_{m\beta}(q) = q$ and since $\phi_{m\beta}$ is an isometry. Thus

$$(7) \quad d(q, \phi_r(a)) \leq \rho + d(\phi_{m\beta}(a), \phi_r(a)).$$

Since ϕ_r is an isometry, we have

$$(8) \quad d(\phi_r(a), \phi_r(q)) = d(q, a) = \rho.$$

We observe that (7) implies $\phi_r(a)$ lies in a closed ball about $\phi_r(q)$ of radius $\rho + d(\phi_{m\beta}(a), \phi_r(a))$. And (8) implies that $\phi_r(a)$ lies in a closed ball about $\phi_r(q)$ of radius ρ . But by (6), these two balls have empty intersection. Hence we have that (1) is true.

From Theorem 1 we obtain immediately the following

Theorem 2. *Suppose X is a Killing vector field on M and q is a critical point of $f = g(X, X)$ such that $f(q) \neq 0$. Suppose the orbit of X through q is closed. If p is a zero point of X , then $d(p, q) \geq D/2$.*

In particular, this theorem gives a lower bound for the distances between zero points of X and orbits of X which are nontrivial closed geodesics. Moreover, the lower bound depends only on the metric and not on the vector field X . To show that it cannot be improved, consider the following example: Let M be S^2 with the usual metric. Let X be the Killing vector field whose flow is a 1-parameter group of rotations about an axis through the north and south poles. Then any point q on the equator is a critical point of $f = g(X, X)$ such that $f(q) \neq 0$ and the north pole N is a zero point of X . D in this case is half of the

circumference of S^2 . Thus we have $d(N, q) = D/2$.

Under additional assumptions on M we can give a lower bound in terms of curvature:

Theorem 3. *Suppose M is a compact, connected Riemannian manifold of even dimension. Suppose the sectional curvatures K_σ satisfy $0 < K_\sigma \leq \lambda$ for all tangent planes σ . Let X be a Killing vector field on M . Then the distance from any zero point of X to any orbit of X which is a nontrivial closed geodesic is always $\geq \pi/4\sqrt{\lambda}$. If in addition we assume M is orientable or simply connected (which are equivalent assumptions), then this distance is $\geq \pi/2\sqrt{\lambda}$.*

Proof. By theorems of Klingenberg [1, pp. 227 and 230], $D \geq \pi/2\sqrt{\lambda}$ ($D \geq \pi/\sqrt{\lambda}$ if M is orientable). The result now follows immediately from Theorem 2.

As a corollary of Theorem 2 we have the following criterion for the zero set of a Killing vector field to be empty in terms of the distribution throughout M of orbits which are nontrivial closed geodesics.

Theorem 4. *Suppose X is a Killing vector field on M with the property that for any point a in M there exists an orbit γ_a of X which is a nontrivial closed geodesic such that the distance from a to γ_a is $< D/2$. Then the zero set of X is empty.*

2. Isolated zero point of X . Let X be a Killing vector field on M with an isolated zero point at p . Then p is a critical point of the function $f = g(X, X)$. We recall that to f there is associated a symmetric bilinear functional f_{**} on $T_p M$ called the Hessian of f at p . The index of f at p is the maximal dimension of a subspace of $T_p M$ on which f_{**} is negative definite. The critical point p is nondegenerate if the nullity of f_{**} on $T_p M$ is zero.

Lemma. *p is a nondegenerate critical point of index zero of the function $f = g(X, X)$.*

Proof. Since X is Killing and p is an isolated zero point, there exists [3, p. 64] a coordinate neighborhood U of p with local coordinates $\{x^1, \dots, x^n\}$ such that (i) $x^i(p) = 0$, $i = 1, \dots, n$, (ii) if $X = \sum_{i=1}^n \xi^i(\partial/\partial x^i)$ on U , then $\xi^i(p) = 0$ and the matrix with (i, j) entry $(\partial \xi^i / \partial x^j)|_p$ is of the form

$$\begin{bmatrix} A_1 & & & & 0 \\ & A_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & A_{n/2} \end{bmatrix}$$

where

$$A_k = \begin{bmatrix} 0 & \theta_k \\ -\theta_k & 0 \end{bmatrix}, \quad \theta_k \neq 0, \quad k = 1, \dots, n/2,$$

and

(iii) if $g = \sum_{i,j} a_{ij} dx^i \otimes dx^j$, then $a_{ij}(p) = \delta_{ij}$. Thus we have $f = \sum_{i,j=1}^n \xi^i \xi^j a_{ij}$.

Let $v = \sum_i v^i (\partial/\partial x^i)|_p$ and $w = \sum_i w^i (\partial/\partial x^i)|_p$ be two vectors in $T_p M$. Extend these to vector fields $\tilde{v} = \sum_i v^i (\partial/\partial x^i)$ and $\tilde{w} = \sum_i w^i (\partial/\partial x^i)$ on U . Then by definition $f_{**}(v, w) = \tilde{v}_p(\tilde{w}(f))$. By a direct computation we see that

$$f_{**}(v, w) = 2\{\theta_1^2(v^1 w^1 + v^2 w^2) + \theta_2^2(v^3 w^3 + v^4 w^4) + \dots + \theta_{n/2}^2(v^{n-1} w^{n-1} + v^n w^n)\}.$$

From this it is obvious that f_{**} is positive definite on all of $T_p M$ and the nullity of f_{**} is zero. Hence we have the result.

Theorem 5. *There exists a local coordinate neighborhood U of p with local coordinates $\{x^1, \dots, x^n\}$ such that with respect to these coordinates, $f = (x^1)^2 + \dots + (x^n)^2$ on U .*

Proof. This is an immediate consequence of the previous lemma and the Morse lemma [4, p. 6].

We have that the following situation prevails near p :

Theorem 6. *There exists a connected open neighborhood U of p satisfying*

- (i) $U - \{p\}$ is a disjoint union of hypersurfaces of M , each of them diffeomorphic to S^{n-1} . (n (even) is the dimension of M .) We call these hyperspheres.
- (ii) The function f is constant on each hypersphere.
- (iii) X is tangent to each hypersphere.
- (iv) Restricted to each hypersphere, the length of X is constant.

Proof. (i) and (ii) follow from Theorem 5. (iii) follows from the fact that f is constant along each integral curve of X , and hence any integral curve of X which meets $U - \{p\}$ lies in one of the hyperspheres. (iv) comes from (ii) and (iii).

Next we consider the question: How far can the neighborhood U be extended so that the conditions of Theorem 6 continue to hold?

Theorem 7. *There exists a connected open neighborhood V of p which contains the neighborhood U of Theorem 6 and which also satisfies conditions (i)–(iv). Moreover, $\bar{V}$ meets an orbit of X which is a nontrivial geodesic.*

Proof. Let Y be the vector field $\text{grad } f / \|\text{grad } f\|^2$ defined on the open submanifold $M - B$ of M , where B denotes the set of critical points of f . Y is of

course nowhere zero on $M - B$. We remark that the orbits (as point sets) of Y are the same as the orbits of $\text{grad } f|_{M-B}$. These orbits are perpendicular to the level surfaces of f . Let $r > 0$ be sufficiently small so that the level surface $f = r$ has a nonempty intersection S with the neighborhood U of Theorem 5. Then S is a hypersphere. Let $t \rightarrow \alpha_t(v)$ be the integral curve of Y through an arbitrary point v of S with $\alpha_0(v) = v$. Then $d\alpha_t(v)/dt = \text{grad } f / \|\text{grad } f\|^2$. Hence we have

$$\frac{df(\alpha_t(v))}{dt} = g\left(\text{grad } f, \frac{\text{grad } f}{\|\text{grad } f\|^2}\right) = 1.$$

Thus we have that

$$(1) \quad f(\alpha_t(v)) = f(\alpha_0(v)) + t = r + t$$

and $\alpha_t(v)$ is defined for t in an interval $(-r, \epsilon_v)$. Let $\epsilon = \inf_{v \in S} \epsilon_v$. Then there exists a family of local diffeomorphisms $\{\phi_t\}_{t \in (-r, \epsilon)}$, each defined on a neighborhood of S . For v in S , $\phi_t(v) = \alpha_t(v)$. Thus $f(\phi_t(v)) = r + t$ for any v in S by (1). Hence f has the constant value $r + t$ on $\phi_t(S)$ and, since ϕ_t is a diffeomorphism, $\phi_t(S)$ is a hypersphere. Now let $V = \{p\} \cup \bigcup_{t \in (-r, \epsilon)} \phi_t(S)$. Then $V \supseteq U$ and V satisfies conditions (i)–(iv).

Now we show that $\bar{V} - \{p\} \cap B \neq \emptyset$. For suppose the contrary. Let A be the open ball about p whose boundary is S . Let v be an arbitrary point of S . Consider the curve $[0, \epsilon) \rightarrow M: t \rightarrow \alpha_t(v)$. This curve is contained in the compact set $\bar{V} - A \subset M - B$. We have that $\|Y\|$ is bounded from above on this compact set. This means that the length of the curve $[0, \epsilon) \rightarrow M: t \rightarrow \alpha_t(v)$ is finite, where the length of this curve is defined by $\lim_{t \rightarrow \epsilon} \int_0^t \|Y(\alpha_s(v))\| ds$. It then follows easily that $\lim_{t \rightarrow \epsilon} \alpha_t(v)$ exists. Call this limit p_v . Then p_v is in $\bar{V} - A$.

Let ρ be the distance from B to $\bar{V} - A$. It is clear that the integral curve of Y starting at p_v goes on for a distance (measured along the integral curve) of at least $\rho/2$. Now let $C = \{q \in M \mid d(q, \bar{V} - A) \leq \rho/2\}$. Then C is compact and $C \subset M - B$. Thus $\|Y\|$ is bounded from above on C . This implies there exists $\eta > 0$ such that the curve $[0, \epsilon) \rightarrow M: t \rightarrow \alpha_t(v)$ may be extended to $[0, \epsilon + \eta) \rightarrow M: t \rightarrow \alpha_t(v)$ and η is independent of v in S . This contradicts the definition of ϵ . Thus $\bar{V} - \{p\} \cap B \neq \emptyset$. From the construction of V it is obvious that if q is a point in $\bar{V} - \{p\} \cap B$, then $f(q) \neq 0$. Thus the orbit of X through q is a non-trivial geodesic meeting V . This concludes the proof.

To show that in general the neighborhood V cannot be further extended in such a way that conditions (i)–(iv) still hold, we need only consider the projective plane with the Killing vector field induced by the vector field X on S^2 given in the example after Theorem 2.

According to Theorem 7, it is reasonable to regard the distance d from p to the nearest orbit of X which is a nontrivial geodesic as a measure of the "size" of the neighborhood V . In case all the orbits of X are closed, we have by Theorem 2 a lower bound for d which is independent of X : namely, $d \geq D/2$. Moreover, a manifold having a Killing vector field with an isolated zero point is even dimensional [3, p. 63]. Thus if the sectional curvatures K_σ of M satisfy $0 < K_\sigma \leq \lambda$, we have by Theorem 3 that $d \geq \pi/4\sqrt{\lambda}$, again assuming all orbits of X are closed.

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